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ON FREEZING OF THE BOUNDARY BETWEEN GROUNDS SATURATED WITH SOLUTIONS OF DIFFERENT TEMPERATURE AND CONCENTRATION

A. G. Egorov

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We study the dynamics of phase transitions at the contact of two porous half-spaces. At the initial time, one of these half-spaces has a temperature below zero. Its pores are partially filled with ice and partially with a concentrated solution which is in thermodynamic equilibrium with the ice at the given temperature. The other half-space is free from ice and contains a warmer and less concentrated solution.

The presence of a two-phase zone — the region in which ice and the solution coexist — is responsible for the specific features of the process considered. This process is described using the approach of [1, 2], which is based on the mechanics of multiphase media. It was used in studies of the thawing or freezing of porous grounds [1–4].

In the situation considered, the picture of phase transitions is complicated, because it includes not only thawing of ice in the initially "cold" part of the space, but, under certain conditions, freezing of ground near the boundary of the half-spaces. Freezing means the occurrence of a zone with an elevated ice content in pores. On complete freezing, ice completely occupies the pores. The main goal of the present work is to obtain criteria of formation for the corresponding zones in terms of initial temperatures and concentrations.

From a practical viewpoint, the interest in the given problem is caused by the problem of the stability of the ice-rock barriers produced in permafrost [5] by sequential pumping of a concentrated brine and fresh water. In addition, the results obtained can also be useful in studies of other processes in frozen grounds. In particular, they point to one of the possible mechanisms of formation of naturally occurring ice lens.

1. Formulation of the Problem. At the initial moment, the porous space is divided into two parts by the plane $x = 0$. At $x > 0$, one part of the pores is occupied by ice, and the other is occupied by a concentrated solution which is in thermodynamic equilibrium with ice. At $x < 0$, the pores are completely filled with a warmer and less concentrated solution.

The heat-and-mass transfer processes that occur in time in the system considered can be described [1, 4] by the equations

$$m\,(\rho_w - \rho_i)\,\frac{\partial \mu}{\partial t} + \rho_w\,\frac{\partial V}{\partial x} = 0; \tag{1.1}$$

$$m\,\frac{\partial \mu c}{\partial t} + \frac{\partial cV}{\partial x} = \frac{\partial}{\partial x}\,D\mu\,\frac{\partial c}{\partial x}; \tag{1.2}$$

$$\frac{\partial}{\partial t}\,(C_0(T - T_f) + \rho_i m L\mu) + C_w\,\frac{\partial}{\partial x}\,(V(T - T_f)) = \frac{\partial}{\partial x}\,\lambda_0\,\frac{\partial T}{\partial x}; \tag{1.3}$$

$$\mu \in H(T + \gamma c - T_f); \tag{1.4}$$

$$C_0(\mu) = (1 - m)C_m + m(1 - \mu)C_i + m\mu C_w, \quad \lambda_0(\mu) = \lambda_i(1 - \mu) + \lambda_w\mu. \tag{1.5}$$

Here t is time, x is the spatial coordinate, m is the porosity, μ is the moisture content (the portion of pores occupied by the solution), V is the filtration rate, c and T are the concentration and temperature of the solution, L is the latent heat of melting, D is the diffusivity of the salt in water, $T_f = 0$ is the temperature of phase transition for pure water, C_0 and λ_0 are the volume heat capacity and heat conductivity of the

Chebotarev Institute of Mathematics and Mechanics of the Kazan' State University, Kazan' 420008.
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porous bed, C_m, C_i, and C_w are the volume heat capacities of the frame, ice, and water, λ_w and λ_i are the heat conductivities of the porous bed with the pores completely filled with water and ice, ρ_w and ρ_i are the densities of water and ice, and γ is the cryoscopic constant.

Relation (1.4) indicates that μ lies on the plot of the corresponding Heaviside function: $0 < \mu < 1$ with satisfaction of the condition $T = T_f - \gamma c$ of local thermodynamic equilibrium, $\mu = 1$ (no ice) at higher temperatures, and $\mu = 0$ (the pores are completely occupied by ice) at low temperatures.

The initial conditions are of the form

$$\begin{aligned} &t = 0,\ x > 0: \quad \mu = \mu_+, \quad c = c_+, \quad T = T_+ = T_f - \gamma c_+ \qquad (0 < \mu_+ < 1), \\ &t = 0,\ x < 0: \quad \mu = \mu_- = 1, \quad c = c_-, \quad T = T_- \qquad (T_- > T_f - \gamma c_-). \end{aligned} \tag{1.6}$$

Let the filtration rate obey the boundary condition

$$-V_+/V_- = \zeta \geqslant 0, \tag{1.7}$$

where ζ is the given constant, and V_+ and V_- are the filtration rates at $x \to +\infty$ and $x \to -\infty$, respectively.

Taking into account that the difference $V_+ - V_-$ is equal to the mass unbalance that occurs as a result of phase transitions [see (1.1)],

$$V_+ - V_- = Q(t) = -m\,\frac{\rho_w - \rho_i}{\rho_i} \int\limits_{-\infty}^{+\infty} \frac{\partial \mu}{\partial t}\, dx,$$

we can rewrite relation (1.7) in one of the following forms: $V_- = -Q/(1+\zeta)$ and $V_+ = \zeta Q/(1+\zeta)$.

The nonnegativeness of ζ means the prohibition of "through" fluid flow through the zone of phase transitions: everywhere at infinity, the fluid moves either to this zone (for $Q < 0$) or from it (for $Q > 0$).

In what follows it will be shown that the main characteristics of the process considered depend only slightly on the choice of ζ. As this quantity we shall use the ratio of the permeabilities of the media for $x = \pm\infty$ unless otherwise specified. Permeability in this case is assumed [1] to be proportional to $\mu^{2/3}$, so that $\zeta = \mu_+^{2/3}$.

The formulated problem (1.1)–(1.7) admits a self-similar solution. We introduce the corresponding self-similar variable

$$\xi = x/2\sqrt{at} \qquad [a = \lambda_w/C_w^0, \quad C_w^0 = (1-m)C_m + mC_w]$$

and normalize $(T - T_f)$, c, and V to

$$T^0 = \frac{\rho_s m L}{C_w^0}, \qquad \frac{T^0}{\gamma}, \qquad m\,\frac{\rho_w - \rho_i}{\rho_w}\sqrt{\frac{a}{t}},$$

retaining the previous notation [T now denotes $(T - T_f)/T^0$].

The original problem reduces to solving the following system of ordinary differential equations:

$$-\xi\mu' + V' = 0; \tag{1.8}$$

$$-\xi(\mu c)' + \varepsilon_v (Vc)' = (1/2)\varepsilon_d(\mu c')'; \tag{1.9}$$

$$-\xi(\alpha T + \mu)' + \varepsilon_m (VT)' = (1/2)(\beta T')'; \tag{1.10}$$

$$\mu \in H(c + T) \tag{1.11}$$

The boundary conditions are

$$\begin{gathered} \mu(\infty) = \mu_+, \quad \mu(-\infty) = 1, \quad V(\infty)/V(-\infty) = -\zeta, \quad c(\infty) = -T_+, \\ T(\infty) = T_+, \quad c(-\infty) = c_-, \quad T(-\infty) = T_-, \end{gathered} \tag{1.12}$$

where

$$\alpha(\mu) = 1 - \alpha_0(1-\mu); \quad \beta(\mu) = 1 + \beta_0(1-\mu); \quad \varepsilon_v = (\rho_w - \rho_i)/\rho_w;$$

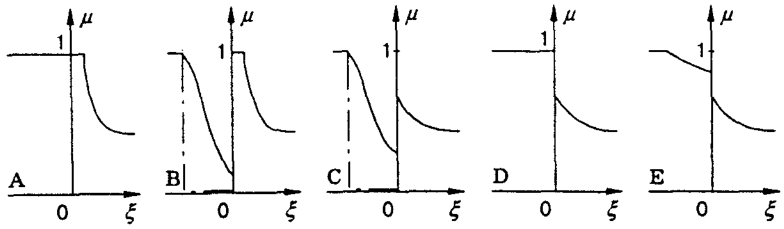

Fig. 1

$$\varepsilon_d = D/a; \quad \varepsilon_m = \varepsilon_v m C_w / C_w^0; \quad \alpha_0 = m(C_w - C_i)/C_w^0; \quad \beta_0 = (\lambda_i - \lambda_w)/\lambda_w.$$

In typical situations, the dimensionless parameters of the problem are small:

$$\varepsilon_v,\ \alpha_0,\ \beta_0 \sim 10^{-1}, \quad \varepsilon_m \sim 10^{-2}, \quad \varepsilon_d \sim 10^{-2}\text{–}10^{-3}. \tag{1.13}$$

Hence, it is reasonable to consider first a simple model in which these parameters in (1.8)–(1.10) are set equal to zero. As can be seen from the following, this model describes well a number of qualitative aspects of the process studied, and, in many cases, it gives good quantitative agreement with the general model (1.8)–(1.12).

2. Analysis of the simple Model. From Eq. (1.9) and boundary conditions (1.12) it follows that the amount of admixture μc per unit pore volume is the known piecewise-constant function of ξ:

$$\mu c = \theta(\xi) = \begin{cases} \theta_+ = \mu_+ c_+, & \xi > 0, \\ \theta_- = \mu_- c_-, & \xi < 0. \end{cases}$$

Hence Eq. (1.11) takes the form $\mu \in H(\theta/\mu + T)$.

Solving this equation for μ, we find

$$\mu = \mu(T, \theta) = \begin{cases} 1, & T \geqslant -\theta, \\ -\theta/T, & T \leqslant -\theta. \end{cases} \tag{2.1}$$

In this case, Eq. (1.10) is written in the form

$$-2\xi(T + \mu(T, \theta(\xi)))' = T'' \tag{2.2}$$

and serves to find T. It can be viewed as an ordinary heat-conduction equation with the known dependence of the heat capacity k on temperature and the spatial coordinate:

$$-2\xi k(T, \xi) T' = T'', \quad k = \begin{cases} 1, & T \geqslant -\theta(\xi), \\ 1 + \theta/T^2, & T \leqslant -\theta(\xi). \end{cases}$$

The solution of the corresponding problem decreases monotonically from T_- (at $\xi = -\infty$) to T_+ (at $\xi = \infty$). According to (2.1), the moisture content μ in this case also decreases monotonically in two-phase zones. The number of such zones and the behavior of the function $\mu(\xi)$ depend on the type of relationship between the three quantities: θ_+, θ_-, and $\theta_* = -T(0)$.

It is easy to verify that the five possible variants of their mutual arrangement ($\theta_- > \theta_*$, $\theta_- < \theta_* < \theta_+$, $\theta_- < \theta_+ < \theta_*$, $\theta_+ < \theta_* < \theta_-$, and $\theta_+ < \theta_- < \theta_*$) correspond to five different modes of behavior of μ, shown schematically in Fig. 1 (A–E).

The value of $T(0)$ depends on θ_-, θ_+, T_-, and T_+. Other parameters fixed, it increases with increase in T_-. Hence, it is clear that, with increase in temperature T_- from T_+ to ∞, the different modes of behavior of the moisture content occurs in the sequence C → B → A for $\theta_- < \theta_+$ and E → D → A for $\theta_+ < \theta_-$.

Regimes B and C are of greatest interest. They are characterized by the formation of negative ξ near the zero of the zone of elevated ice content in the pores. Extending the meaning of the well-known term, we shall call such a zone a lens. It follows from (2.1) that, with decrease in c_-, the ice content in the lens increases. In the limit $c_- \to 0$, we have $\mu \to 0$ in the lens. This situation is shown by the dot-and-dashed

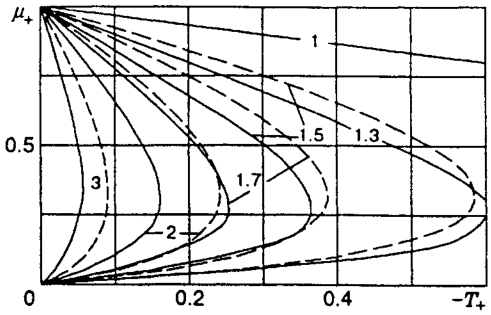

Fig. 2

curves in Fig. 1. The corresponding lens will be referred to as impermeable, which implies the absence of a liquid phase in it.

We examine the case $c_- = 0$ in greater detail. Here $\theta_- = 0$ and, hence, by virtue of the positiveness of θ_+, the modes of behavior of mixture content can occur only in the sequence C $\to$ B $\to$ A as T_- increases. In this case, the most important is the question of the critical temperature T_-^* at which, at fixed μ_+ and T_+, transition between regimes A and B occurs, i.e., an impermeable lens is formed.

To find the critical temperature as a root of the equation $\theta_*(T_-^*, \mu_+, T_+) = 0$, it is necessary to solve the differential equation (2.2) repeatedly. It was solved numerically on a grid which was made finer near $\xi = 0$ by the upper relaxation method. Because of the rapid decay of T and μ as $|\xi| \to \infty$, we used a finite calculation interval and transferred boundary conditions of the form (1.12) from infinity to its ends. The choice of the segment $|\xi| < 4$ as the calculation interval was adequate in the calculations.

The calculation results are shown in Fig. 2 (dashed curves) as level lines of the function $U_1(T_+, \mu_+) = -T_-^*/T_+$. This ratio increases monotonically with increase in T_+ from unity at $T_+ = -\infty$ to infinity at $T_+ = 0$. The dependence of the critical temperature on μ_+ is nonmonotonic. The minimum value of U_1, equal to unity, is reached at the points $\mu_+ = 1$ and $\mu_+ = 0$, and the maximum value is reached at the point $\mu_+^{\max}$. The quantity $\mu_+^{\max}$ practically does not depend on T_+ and is equal to approximately 0.33.

Note that an impermeable lens also occurs in the situation where, initially, there was no ice in the porous bed ($\mu_+ = 1$). Moreover, the necessity of formation of an impermeable lens is most understandable precisely in this case.

Indeed, if the phase transition "water–ice" did not proceed there, the temperature of the medium would be distributed by the law $T = T_+ - (1/2)(T_+ - T_-)\mathrm{erfc}(\xi)$ and take the value $T_0 = (T_+ + T_-)/2$ at $\xi = 0$. The dependence $c(\xi)$ would be a step function and coincide with zero for $\xi < 0$ and with $-T_+$ for $\xi > 0$. At the same time, by virtue of the one-phase solution of the problem, the inequality $T(\xi) > -c(\xi)$ should be satisfied everywhere. The point $\xi = 0$ is critical for this inequality. The one-phase condition $T_0 > -c(-0)$ can be satisfied at this point only at sufficiently high T_-, i.e., at $T_- > T_-^* = -T_+$. At lower T_-, the curves of $T(\xi)$ and $-c(\xi)$ intersect at negative values of ξ, and this indicates the necessity of formation of a lens there.

Similar reasoning can be used for the general formulation of problem (1.8)–(1.12). The role of diffusion in this case reduces to spreading the step in $c(\xi)$, and the role of convection reduces to transferring this step to the right (at positive V). From geometrical considerations it is clear that the first will prevent the formation of a lens, and the second will facilitate this.

3. Analysis of the General Model. Problem (1.8)–(1.12) in the general formulation was solved numerically on a grid that was made finer near $\xi = 0$. As before, the boundary conditions (1.12) were transferred from infinity at the ends $\xi = \pm 4$ of the calculation interval. An iterative method with splitting of the problem into physical processes was used. In each iteration, proceeding from the values of V and $\theta = \mu c$ and using the upper relaxation method, we first sought the functions T and μ as a solution of the problem (1.10) and (2.1). After that, the values of V were determined by integrating (1.8), and the values of c were determined by the sweep method from Eq. (1.9) with the functions V and μ calculated in this iterative step.

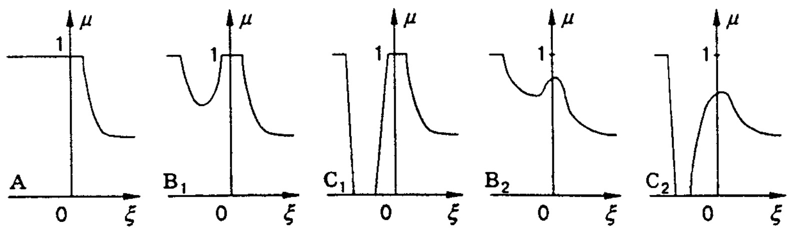

Fig. 3

The calculations were performed, as a rule, on a grid with 100 points, a third of which fall into the zone of lens. The results were tested on grids that had 2 and 4 times the number of points.

The calculation results depend only slightly on the values of α_0, β_0, and ε_m with the orders of magnitude determined in (1.13). The latter indicates that convective heat transfer in the processes studied can be ignored compared with conduction. The convective transfer of the admixture, however, is significant because of the weak diffusion mechanism.

Thus, the difference of the general formulation of the problem from the simplified formulation consists primarily in taking into account transfer (convective and diffusion) of the admixture. Based on the calculations performed, we consider in greater detail the effect of this on the lens-formation process.

As will be shown below, the filtration rate in the lens zone is always positive. Thus, the weakly concentrated salt solution is pushed here to the low-temperature region, and this leads to freezing of the porous bed. This effect works at nonzero values of c_-. Therefore, an impermeable lens can form there for $c_- > 0$ as well, in contrast to the simple model. The effect of convective mass transfer on the values of μ, c, and T is confined within the zone of the lens (low μ). Beyond the lens zone, neglect of convection does not lead to any marked change in the desired functions. This is easy to understand by comparing the first two terms of Eq. (1.9).

Furthermore, it is evident from this equation that the effect of the diffusion process is significant at small ξ, i.e., again in the lens zone. The admixture flow directed to this zone from the zone of high concentrations decreases the local thawing point in it, thus preventing the growth of the lens. This effect leads, in particular, to the fact that, at $c_- = 0$, not every lens is impermeable, as in the simple model.

Figure 3 shows schematically the possible modes of behavior of the function $\mu(\xi)$ in the case $\theta_- < \theta_+$. They correspond to the regimes A, B, and C of the simple model. With increase in T_-, the function μ changes in the sequence $C_1 \to B_1 \to A$ (low T_+) or $C_2 \to B_2 \to A$ (high T_+). In both cases, there are two critical values of T_-: (1) T_-^1 corresponds to the occurrence of a local minimum of the function $\mu(\xi)$ (transition $B_1 \to A$ or $B_2 \to A$), i.e., the onset of formation of a lens; (2) T_-^2 corresponds to the occurrence of a point at which μ vanishes (transition $B_1 \to C_1$ or $B_2 \to C_2$), i.e., the occurrence of an impermeable lens. The values of the critical temperatures depend on μ_+, T_+, and c_- and reach their minimum at $c_- = 0$.

For this case ($c_- = 0$), Fig. 2 (solid curves) and Fig. 4 show the level lines of the functions $U_1(T_+, \mu_+) = -T_-^1/T_+$ and $U_2(T_+, \mu_+) = -T_-^2/T_+$. In the calculations we assumed that $\alpha_0 = \beta_0 = \varepsilon_m = 0$ and $\varepsilon_d = 3 \cdot 10^{-3}$.

The value of U_1 increases infinitely at $T_+ \to 0$ and $0 < \mu < 1$. At $\mu_+ = 0$, it can be equal to unity, and, at $\mu_+ = 1$, it can be somewhat smaller than unity. The corresponding value can be determined using the same line of reasoning as that in Sec. 2 for a similar situation. As $\varepsilon_d \to 0$, it is approximately equal to $1 - 2\sqrt{\varepsilon_d}$, and this is 0.89 for the given ε_d. The special level line $U_1 = 1$ issues out of the point $\mu_+ = 1$, $T_+ = 0$ and is closed by the axis $\mu_+ = 0$ for $T_+ \to -\infty$.

It can be seen from Fig. 2 that the critical value of T_-^1 agrees qualitatively well with the critical value of T_-^* for the simplified model. A certain decrease in T_-^1 compared with T_-^* is explained by the action of the diffusion transfer of the admixture, while convection, which is significant only for small μ, has not yet manifested itself.

The function $U_2(\mu_+, T_+)$ reaches a maximum value ($\approx$1.87) at the point $\mu_+ \approx 0.45$, $T_+ \approx -0.06$. The level lines of this function for $U_2 > 1$ are closed, and, for $U_2 < 1$, they originate from the straight line $\mu_+ = 1$

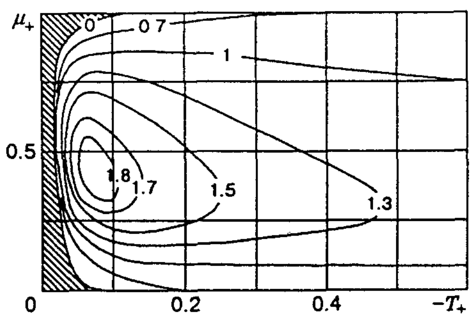

Fig. 4

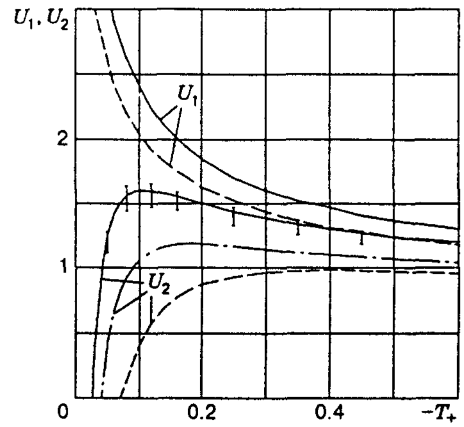

Fig. 5

and approach asymptotically the axis $\mu_+ = 0$. In the dashed region in Fig. 4, U_2 is not defined. In this region, the second critical temperature is not reached, and, hence, an impermeable lens is formed at none of the values of T_-. In particular, it cannot occur at $T_+ > T_+^{\min} \approx -0.018$ irrespective of the values of μ_+ and T_-.

Of the two critical temperatures U_1 and U_2, the second decreases much more rapidly with increase in c_-. Thus, with increase in c_- from zero to 0.01, the value of U_1 changes by less than 7%, whereas $T_+^{\min}$ decreases from -0.018 to -0.21. This value reaches -0.6 even at $c_- = 0.03$. For a porosity $m = 0.3$ and a solution of NaCl, the indicated values correspond to a concentration of 4 g/liter and a temperature of -15°C. Thus, it is clear that the necessary condition of formation of an impermeable lens is the contact of the concentrated brine with practically pure water.

With increase in the diffusivity ε_d, the critical parameters behave in the same manner. This is illustrated in Fig. 5, which shows the values of the functions $U_1(T_+)$ and $U_2(T_+)$ at $\varepsilon_d = 0.003$ (solid curves) and $\varepsilon_d = 0.01$ (dashed curves) calculated for $\mu_+ = 0.3$ and $c_- = 0$.

We examine the lens-formation process with variation in the value of ζ, which characterizes the ratio of the liquid flows from infinity into the phase-transition zone. The limiting values of ζ (0 and ∞) are reached in the absence of liquid inflow from the frozen and thawing zone, respectively. The intermediate values of ζ correspond to zero filtration rate at the final point of the region. In any case, the difference of the values of $V(\xi)$ from zero is associated only with phase transitions [see (1.8)], and its behavior is governed by the type of $\mu(\xi)$. It follows from Fig. 3 and Eq. (1.8) that the value of V increases at negative ξ, reaches a maximum in the lens zone, and then decreases. This leads to a positive filtration rate in the lens region: otherwise, $V(\xi)$ would have been negative everywhere and zero would not have reached.

The smallness of V does not permit convection to affect significantly the main characteristics of the processes beyond the range of small values of μ. It is not surprising, therefore, that the first critical temperature practically does not depend on the choice of ζ. It is somewhat unexpected, however, that the second critical temperature also depends only slightly on ζ. This can be seen in Fig. 5, which shows the spread of U_2 for the limiting values of ζ; the minimum corresponds to $\zeta = \infty$, and the maximum corresponds to $\zeta = 0$.

This fact requires explanations, because, in the absence of convection ($V \equiv 0$), the second critical temperature is not defined at all, and an impermeable lens is not formed. In this case, with decrease in T_- in the lens zone, the value of $\mu(\xi)$ becomes smaller and smaller, but never vanishes. The value of T_- for which μ reached a given small value can serve as an analog of the second critical temperature. The function U_2 that corresponds to the second temperature and calculated for the threshold value $\mu_+ = 0.03$ is shown in Fig. 5 by the dot-and-dashed curve.

For large values of $-T_+$, its departure from similar curves constructed with allowance for convective mass transfer is small. This indicates that the main contribution to the formation of an impermeable lens here is associated with diffusion. With increase in T_+, the difference of the indicated curves increases. Convection plays an ever greater role. But, simultaneously, the value of the velocity $V^{\max}$ in the lens zone depends more and more slightly on the choice of ζ, and this is associated with an increase in the portion of internal flows in the phase-transition zone compared with liquid inflow to this zone from infinity: melting and freezing in various portions of the zone compensate one another. The ratio $|V^{\max} - V_+|/|V_+ - V_-|$, which does not depend on the choice of ζ and characterizes this phenomena, calculated for the second critical temperature for $T_+ = -0.05$ is equal to 1.4, and, for $T_+ = -0.5$, it is only 0.27.

The weak dependence of the desired characteristics on the choice of ζ justifies to a certain degree the value of ζ adopted in the present paper.

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